Week 11 Fourier Series (Reference: An introduction to Hilbert Space, N. Young)

Last week: $\mathbb{F} = \mathbb{R}$ (similar for $\mathbb{C}$)

$\{1, x, x^2, \cdots\}$ Basis for $P(\mathbb{R}) \subseteq L^2[-1,1]$

$\downarrow$ G.S process

$\{e_0, e_1, e_2, \cdots\}$ orthonormal basis

where e_n is a deg n polynomial

Under L^2-norm: $\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,dx$

$$\underbrace{P(\mathbb{R}) \subseteq C[-1,1]}_{\text{dense}} \underbrace{\subseteq L^2[-1,1]}_{\text{dense (measure theory)}}$$

Reason: Weierstrass approximation thm:

For $f \in C[-1,1]$, $\exists$ polynomial $p_n \to f$ uniformly

i.e. $\|p_n - f\|_\infty \to 0$

$\Rightarrow p_n \to f$ in L^2-norm

Consequence: $\overline{P(\mathbb{R})} = L^2[-1,1]$

$\therefore \overline{\operatorname{Span}\{e_0, e_1, \cdots\}} = L^2[-1,1]$

$\therefore \forall f \in L^2[-1,1]$

$$f = \sum_{n=0}^{\infty} \langle f, e_n\rangle e_n \quad \text{in } L^2\text{-norm}$$ (Hilbert space theory)

Rmk Everything is same for $\mathbb{F} = \mathbb{C}$

Different types of convergence for functions

Pointwise convergence

$f_n \to f$ pointwisely means $\forall x \ \lim_{n\to\infty} f_n(x) = f(x)$

L^∞-convergence / uniform convergence

$f_n \to f$ uniformly means $\|f - f_n\|_\infty \to 0$

i.e. $\forall \varepsilon > 0, \ \exists N$ s.t. $\forall n \geq N, \forall x, \ |f(x) - f_n(x)| < \varepsilon$

L^2-convergence

$f_n \to f$ in L^2-norm means $\|f - f_n\|_2 \to 0$

Note if $f_n \to f$ uniformly on $[a, b]$, then

$$\|f_n - f\|_2 = \sqrt{\int_a^b |f_n(x) - f(x)|^2 dx} \leq \sqrt{\|f_n - f\|_\infty^2 (b-a)} = \|f_n - f\|_\infty \sqrt{b-a}$$

$\therefore f_n \to f$ in L^2-norm

This time: $\boxed{\text{Fourier Series}}$

Consider $L^2[-\pi,\pi]$, $i=\sqrt{-1}$, $n\in\mathbb{Z}$

$$e_n(x)=\frac{1}{\sqrt{2\pi}}e^{inx} \quad \text{in } L^2[-\pi,\pi]$$

Recall: $e^{i\theta}=\cos\theta+i\sin\theta \qquad \theta\in\mathbb{R}$

$$e_0(x)=\frac{1}{\sqrt{2\pi}} \quad \text{constant function}$$

$$e_1(x)=\frac{1}{\sqrt{2\pi}}(\cos x+i\sin x)$$

$$e_{-1}(x)=\frac{1}{\sqrt{2\pi}}(\cos x-i\sin x)=\overline{e_1(x)}$$

$$e_n(x)=\frac{1}{\sqrt{2\pi}}(\cos(nx)+i\sin(nx))$$

$$e_{-n}(x)=\frac{1}{\sqrt{2\pi}}(\cos(nx)-i\sin(nx))=\overline{e_n(x)}$$

Recall: If $f(x)=a(x)+ib(x)$, where $a(x), b(x)$ are real-valued

$$\int_{-\pi}^{\pi}f(x)\,dx=\int_{-\pi}^{\pi}a(x)\,dx+i\int_{-\pi}^{\pi}b(x)\,dx$$

<u>Prop</u> $\{e_n\}_{n\in\mathbb{Z}}=\{\cdots,e_{-1},e_0,e_1,e_2,\cdots\}$ is orthonormal subset of $L^2[-\pi,\pi]$

<u>Pf</u> $\langle e_m,e_m\rangle=\int_{-\pi}^{\pi}e_m(x)\overline{e_m(x)}\,dx$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{imx}e^{-imx}\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}1\,dx$$

$$=1 \qquad \text{(Exercise)}$$

Similarly, $\langle e_m,e_n\rangle=0$ for $m\neq n$

Q Is $\{e_n\}_{n\in\mathbb{Z}}$ total in $L^2[-\pi,\pi]$

i.e. $\overline{\operatorname{Span}\{e_n\}_{n\in\mathbb{Z}}} = L^2[-\pi,\pi]$?

i.e. $\forall f \in L^2[-\pi,\pi]$

$$\lim_{m\to\infty} \sum_{n=-m}^{m} \langle f, e_n\rangle e_n = f \ ?$$

$\boxed{\text{Answer is Yes!}}$

Strategy:

1. Consider 2π-periodic $f: \mathbb{R} \to \mathbb{R}$

Let $f_m = \sum_{n=-m}^{m} \langle f, e_n\rangle e_n$

Want to show $f_m \to f$ (a little bit hard)

Instead, we consider

$$F_m = \underbrace{\frac{1}{m+1}(f_0 + f_1 + f_2 + \cdots + f_m)}_{\text{arithmetic mean of first } m+1 \text{ terms}}$$

(Easier) Show that $F_m \to f$ uniformly on $\mathbb{R}$

2: By 1 $F_m \to f$ uniformly on $[-\pi,\pi]$

$\Rightarrow F_m \to f$ in $L^2[-\pi,\pi]$

$$\operatorname{Span}\{e_n\} \underset{\text{dense}}{\subseteq} \left\{\begin{matrix} 2\pi\text{-periodic} \\ f: \mathbb{R}\to\mathbb{R} \end{matrix}\right\} \underset{\text{dense}}{\subseteq} C[-\pi,\pi] \underset{\text{dense}}{\subseteq} L^2[-\pi,\pi]$$

$\Rightarrow \overline{\operatorname{Span}\{e_n\}} = L^2[-\pi,\pi]$

Formula for F_m and Fejer kernel

$f: \mathbb{R} \to \mathbb{R}$ is 2π-periodic

i.e. $f(x+2\pi) = f(x) \quad \forall x$

$$f_m = \sum_{n=-m}^{m} \langle f, e_n \rangle e_n$$

$$f_m(y) = \sum_{n=-m}^{m} \left(\int_{-\pi}^{\pi} f(x) \overline{\frac{1}{\sqrt{2\pi}} e^{inx}} dx \right) \frac{1}{\sqrt{2\pi}} e^{iny}$$

$$= \frac{1}{2\pi} \sum_{n=-m}^{m} \int_{-\pi}^{\pi} f(x) e^{in(y-x)} dx$$

y is regarded as constant in integration

$$F_m(y) = \frac{1}{m+1} \sum_{j=0}^{m} f_j(y)$$

$$= \frac{1}{m+1} \sum_{j=0}^{m} \left(\frac{1}{2\pi} \sum_{n=-j}^{j} \int_{-\pi}^{\pi} f(x) e^{in(y-x)} dx \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \underbrace{\frac{1}{m+1} \sum_{j=0}^{m} \sum_{n=-j}^{j} e^{in(y-x)}}_{K_m(y-x)} dx$$

Let $K_m(t) = \frac{1}{m+1} \sum_{j=0}^{m} \sum_{n=-j}^{j} e^{int}$

Defn K_m is called Fejer Kernel

Then

$$\boxed{F_m(y) = \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} f(x) K_m(y-x) dx}_{\text{called convolution of } f \text{ and } K_m}}$$

$f * K_m$

Lemma

$$\boxed{K_m(t) = \frac{1}{m+1} \cdot \frac{\sin^2 \frac{(m+1)t}{2}}{\sin^2 \frac{t}{2}} \quad \text{for } t \neq 2k\pi,\ k \in \mathbb{Z}}$$

Pf

$$(m+1)K_m(t) = \sum_{j=0}^{m} \sum_{n=-j}^{j} e^{int}$$

$$= \sum_{j=0}^{m} \left[1 + \sum_{n=1}^{j} \left(e^{int} + e^{-int} \right) \right]$$

$$= \sum_{j=0}^{m} \left[1 + \sum_{n=1}^{j} 2\cos(nt) \right]$$

$$\sin\frac{t}{2}(m+1)K_m(t) = \sum_{j=0}^{m} \left[\sin\frac{t}{2} + \sum_{n=1}^{j} 2\cos(nt)\sin\frac{t}{2} \right]$$

$$= \sum_{j=0}^{m} \left(\sin\frac{t}{2} + \sum_{n=1}^{j} \left[\sin\left(n+\tfrac{1}{2}\right)t - \sin\left(n-\tfrac{1}{2}\right)t \right] \right)$$

$$= \sum_{j=0}^{m} \left(\sin\frac{t}{2} + \sin\left(j+\tfrac{1}{2}\right)t - \sin\frac{t}{2} \right)$$

$$= \sum_{j=0}^{m} \sin\left(j+\tfrac{1}{2}\right)t$$

$$\left(\sin\frac{t}{2}\right)^2 (m+1)K_m(t)$$

$$= \sum_{j=0}^{m} \sin\left(j+\tfrac{1}{2}\right)t \, \sin\frac{t}{2}$$

$$= \sum_{j=0}^{m} \frac{-1}{2} \left[\cos(j+1)t - \cos jt \right]$$

$$= -\frac{1}{2} \left[\cos(m+1)t - 1 \right]$$

$$= \frac{1}{2} \left[1 - \cos(m+1)t \right]$$

$$= \sin^2 \frac{(m+1)t}{2}$$

Rearrange terms, done

Properties of $K_m(t)$

① $K_m(t) \geq 0$

② $\int_{-\pi}^{\pi} K_m(t)\,dt = 2\pi$

③ For any δ, $0 < \delta < \pi$

$$\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_m(t)\,dt \to 0$$

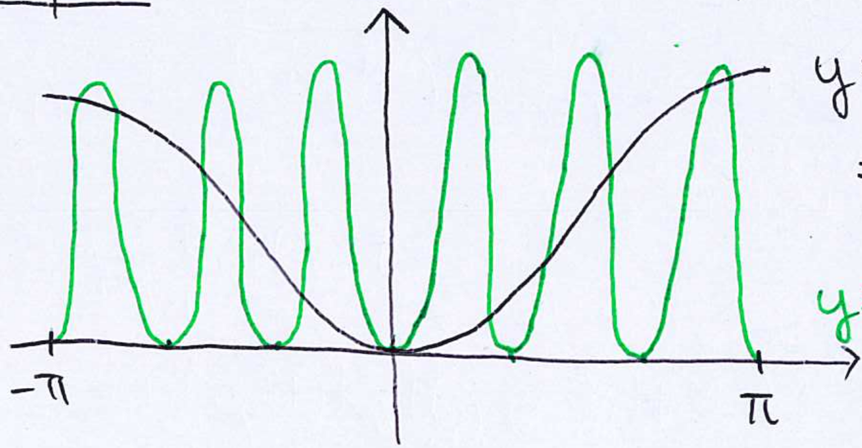
as $m \to \infty$

② + ③

Area $= 2\pi$

concentrated near 0 as $m \to \infty$

Graphs:



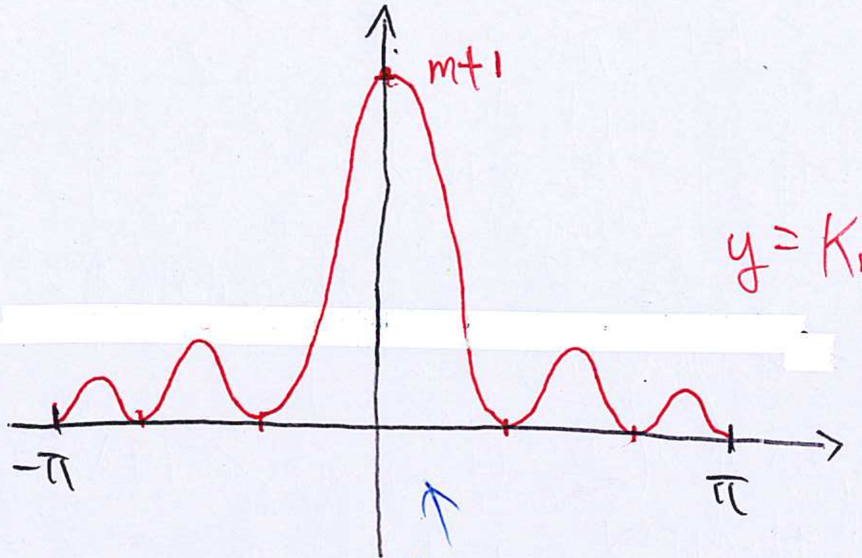
$y = \sin^2 \frac{t}{2}$

$= \frac{1}{2}(1 - \cos t)$

$y = \sin^2 \frac{(m+1)t}{2}$

$m = 5$

$$K_m(t) = \frac{1}{m+1} \frac{\sin^2\left(\frac{m+1}{2} t\right)}{\sin^2 \frac{t}{2}}$$



$y = K_m(t)$, $m = 5$

"Area concentrated near 0"

Pf of Properties

1. Clear from lemma

2. $\int_{-\pi}^{\pi} K_m(t)\,dt$

$$= \frac{1}{m+1}\int_{-\pi}^{\pi}\sum_{j=0}^{m}\sum_{n=-j}^{j} e^{int}\,dt$$

$$= \frac{1}{m+1}\sum_{j=0}^{m}\sum_{n=-j}^{j}\int_{-\pi}^{\pi} e^{int}\,dt$$

$$= \frac{1}{m+1}\sum_{j=0}^{m} 2\pi = 2\pi$$

Note $\int_{-\pi}^{\pi} e^{int}\,dt = \begin{cases} 2\pi, & n=0 \\ 0, & n\neq 0 \end{cases}$

3. $K_m(t) = \frac{1}{m+1}\,\frac{\sin^2\left(\frac{m+1}{2}t\right)}{\sin^2\frac{t}{2}}$

Let $0<\delta<\pi$

For $-\pi\leq t\leq -\delta$ and $\delta\leq t\leq\pi$

$$\sin^2\frac{t}{2} \geq \sin^2\frac{\delta}{2}$$

$$\therefore 0\leq K_m(t)\leq \frac{1}{m+1}\,\frac{1}{\sin^2\frac{\delta}{2}}$$

$$\Rightarrow 0\leq\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)K_m(t)\,dt \leq \frac{2\pi}{m+1}\,\frac{1}{\sin^2\frac{\delta}{2}}$$

When $m\to\infty$, L.H.S $=0$, RHS $\to 0$

Sandwich theorem $\Rightarrow$ 3

Back to F_m

Want to show $F_m \to f$

$$F_m(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\,K_m(y-x)}_{2\pi\text{-periodic}}\,dx$$

$$= \frac{1}{2\pi}\int_{y-\pi}^{y+\pi} f(x)\,K_m(y-x)\,dx$$

$$f(y) = f(y)\,\frac{1}{2\pi}\int_{y-\pi}^{y+\pi} K_m(y-x)\,dx$$

$$= \frac{1}{2\pi}\int_{y-\pi}^{y+\pi} f(y)\,K_m(y-x)\,dx$$

$$F_m(y) - f(y)$$

$$= \frac{1}{2\pi}\int_{y-\pi}^{y+\pi} \underbrace{\big(f(x)-f(y)\big)}_{\substack{\text{small when}\\ x,y \text{ are close}}}\,\underbrace{K_m(y-x)}_{\substack{\text{small when}\\ x,y \text{ are not too close}\\ \text{and } m \gg 0}}\,dx$$

$$|F_m(y) - f(y)|$$

δ is chosen based on f

$$\le \frac{1}{2\pi}\left(\int_{y-\pi}^{y-\delta} + \int_{y+\delta}^{y+\pi} + \int_{y-\delta}^{y+\delta}\right)|f(x)-f(y)|\,K_m(y-x)\,dx$$

Analysis of the three integrals (For details, see reference or next page)

$\Rightarrow$ $F_m(y) - f(y) \to 0$ as $m \to \infty$

Moreover, m is indept of y (ie. uniform convergence)

Details for the uniform convergence of $f_m \to f$ (Optional, not for exam)

Given $\varepsilon > 0$, we want to find m_0 such that $|F_m(y) - f(y)| < \varepsilon \quad \forall y \in \mathbb{R}$ and $m \geq m_0$

Pf. f is continuous on $[-\pi, \pi] \Rightarrow f$ is uniform continuous on $[-\pi, \pi]$

Also, f is 2π-periodic $\Rightarrow \exists M > 0$ such that $|f(x)| < M \quad \forall x \in \mathbb{R}$

Also, $\exists \delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2} \quad \forall x, y \in \mathbb{R}, |x-y| < \delta$

By property ③ of K_m on P7, $\exists m_0$ such that $\forall m \geq m_0$, $\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_m(t)\,dt < \frac{\pi\varepsilon}{2M}$

Hence, $\forall y \in \mathbb{R}$, $m \geq m_0$,

$$|F_m(y) - f(y)| \leq \frac{1}{2\pi}\left(\int_{y-\pi}^{y-\delta} + \int_{y+\delta}^{y+\pi}\right)|f(x) - f(y)|\, K_m(y-x)\,dx + \frac{1}{2\pi}\int_{y-\delta}^{y+\delta}|f(x) - f(y)|\, K_m(y-x)\,dx$$

$$\leq \frac{1}{2\pi}(2M)\left[\left(\int_{y-\pi}^{y-\delta} + \int_{y+\delta}^{y+\pi}\right) K_m(y-x)\,dx\right] + \frac{1}{2\pi}\cdot\frac{\varepsilon}{2}\cdot\int_{y-\delta}^{y+\delta} K_m(y-x)\,dx$$

$$\leq \frac{M}{\pi}\cdot\frac{\pi\varepsilon}{2M} + \frac{1}{2\pi}\,\frac{\varepsilon}{2}\,2\pi = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Summary Consider $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$

① Let $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ $n \in \mathbb{Z}$

then $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal

② Given 2π-periodic continuous $f: \mathbb{R} \to \mathbb{R}$

Define $f_m = \sum_{n=-m}^{m} \langle f, e_n \rangle e_n$

$$F_m = \frac{1}{m+1} \sum_{j=0}^{m} f_j$$

Similar for $f: \mathbb{R} \to \mathbb{C}$

We showed

$F_m \to f$ uniformly on $\mathbb{R}$

$\Rightarrow F_m \to f$ uniformly on $[-\pi, \pi]$

$\Rightarrow F_m \to f$ in $L^2[-\pi, \pi]$

Consider

$$\operatorname{Span}\{e_n\}_{n\in\mathbb{Z}} \underset{\text{dense by ②}}{\subseteq} \{f: \mathbb{R} \to \mathbb{C},\ \text{continuous}, 2\pi\text{-periodic}\}$$

$$\subseteq C[-\pi, \pi]$$

dense (easy to prove)

$$\subseteq L^2[-\pi, \pi]$$

dense (measure theory)

$$\therefore \overline{\operatorname{Span}\{e_n\}_{n\in\mathbb{Z}}} = L^2[-\pi, \pi]$$